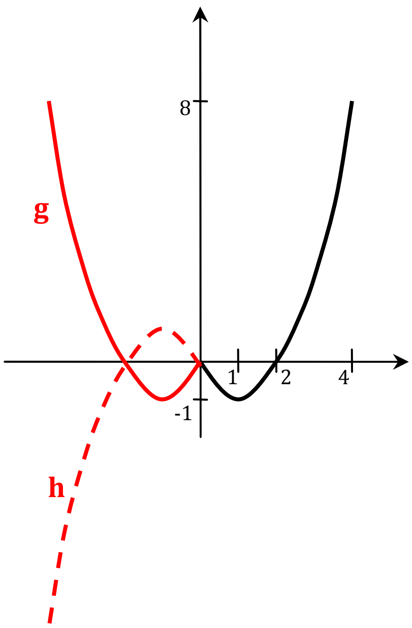
1. Je dána funkce $f(x) = x^2 - 2x$ v intervalu $(0; 4)$. Nakreslete graf jejího sudého rozšíření **g** a lichého rozšíření **h** na intervalu $(-4; 4)$. Pak sestavte kosinovou Fourierovu řadu funkce f v intervalu $(0; 4)$. Vypište koeficienty a_1, a_2.



$D = 4$

$x_{1,2} = \dfrac{2 \pm 2}{2} = \begin{matrix} x_1 = 0 \\ x_2 = 2 \end{matrix}$ – průsečíky s y

$V = \dfrac{-b}{2a} = \dfrac{2}{2} = 1$ – x vrcholu

$f(1) = -1$ – y vrcholu

$2\ell = 8$

$\ell = 4$

$b_n = 0$

$$a_0 = \frac{1}{4}.2.\int_0^4 (x^2 - 2x).dx = \frac{1}{2}.\left[\frac{x^3}{3} - 2.\frac{x^2}{2}\right]_0^4 = \frac{1}{2}.\left(\frac{64}{3} - 16\right) = \frac{8}{3}$$

$u' = 2x - 2 \qquad v = \dfrac{1}{n\frac{\pi}{4}}.\sin n\frac{\pi}{4}x$

$$a_n = \frac{1}{4}.2.\int_0^4 \overbrace{(x^2 - 2x)}^{u}.\overbrace{\cos n\frac{\pi}{4}x}^{v'}.dx =$$

$u' = 2 \qquad v = -\dfrac{4}{n\pi}.\cos n\frac{\pi}{4}x$

$$= \frac{1}{2}.\left\{\frac{4}{n\pi}.\underbrace{\left[(2x-2).\sin n\frac{\pi}{4}x\right]_0^4}_{=0} - \frac{4}{n\pi}\int_0^4 \overbrace{(2x-2)}^{u}.\overbrace{\sin n\frac{\pi}{4}x}^{v'}.dx\right\} =$$

$$= -\frac{2}{n\pi}.\left\{-\frac{4}{n\pi}.\left[(2x-2).\cos n\frac{\pi}{4}x\right]_0^4 + \frac{4}{n\pi}.\int_0^4 \cos n\frac{\pi}{4}x.dx\right\} =$$

$$= -\frac{8}{n^2\pi^2}.\left\{-[6.(-1)^n + 2] + \frac{4}{n\pi}.\underbrace{\left[\sin n\frac{\pi}{4}x\right]_0^4}_{=0}\right\} = \frac{8}{n^2\pi^2}.[6.(-1)^n + 2] =$$

$$= \frac{16}{n^2\pi^2}.[3.(-1)^n + 1]$$

$$a_1 = \frac{16}{1^2\pi^2}.(-3 + 1) = -\frac{32}{\pi^2}$$

$$a_2 = \frac{16}{2^2\pi^2}.(3 + 1) = \frac{16}{\pi^2}$$

$$f(x) \sim \frac{4}{3} + \frac{16}{\pi^2} . \sum_{n=1}^{\infty} \frac{3.(-1)^n + 1}{n^2} . \cos n \frac{\pi}{4} x \quad v\ (0; 4)$$

2. Najděte obecné řešení soustavy diferenciálních rovnic

$$\begin{aligned} y_1' &= -2y_1 + 16y_2 - 4y_3 \\ y_2' &= 2y_2 - y_3 \\ y_3' &= 2y_1 - 4y_3 \end{aligned}$$

$$|A - \lambda E| = \begin{vmatrix} -2-\lambda & 16 & -4 \\ 0 & 2-\lambda & -1 \\ 2 & 0 & -4-\lambda \end{vmatrix} =$$

$$= (-2-\lambda).\begin{vmatrix} 2-\lambda & -1 \\ 0 & -4-\lambda \end{vmatrix} - 16.\begin{vmatrix} 0 & -1 \\ 2 & -4-\lambda \end{vmatrix} - 4.\begin{vmatrix} 0 & 2-\lambda \\ 2 & 0 \end{vmatrix} =$$

$$= (-2-\lambda).(-8 - 2\lambda + 4\lambda + \lambda^2) - 16.(2) - 4.(-4 + 2\lambda) =$$

$$= 16 + 4\lambda - 8\lambda - 2\lambda^2 + 8\lambda + 2\lambda^2 - 4\lambda^2 - \lambda^3 - 32 + 16 - 8\lambda =$$

$$= -\lambda^3 - 4\lambda^2 - 4\lambda = 0$$

$$\lambda^3 + 4\lambda^2 + 4\lambda = 0 \Rightarrow \lambda.(\lambda^2 + 4\lambda + 4) = 0 \Rightarrow \lambda_1 = 0$$

$$D = 16 - 4.1.4 = 0$$

$$\lambda_{2,3} = \frac{-4}{2} = -2$$

Pro $\lambda_1 = 0$:

$$\begin{aligned} -2k_1 + 16k_2 - 4k_3 &= 0 \\ 2k_2 - 1k_3 &= 0 \Rightarrow 2k_2 = k_3 \\ 2k_1 - 4k_3 &= 0 \Rightarrow 2k_1 = 4k_3 \Rightarrow 2k_1 = 8k_2 \Rightarrow k_1 = 4k_2 \end{aligned}$$

Např. :$k_1 = 4, k_2 = 1, k_3 = 2$

$$^1\vec{y} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} . e^{0x}$$

Pro $\lambda_2 = -2$:

$$\begin{aligned} 0k_1 + 16k_2 - 4k_3 &= 0 \\ 4k_2 - 1k_3 &= 0 \Rightarrow 4k_2 = k_3 \\ 2k_1 - 2k_3 &= 0 \Rightarrow k_1 = k_3 = 4k_2 \end{aligned}$$

Např. :$k_1 = 4, k_2 = 1, k_3 = 4$

$$^2\vec{y} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} . e^{-2x}$$

Pro $\lambda_3 = -2$:

$$^3\vec{y} = \begin{pmatrix} a_1x + a_0 \\ b_1x + b_0 \\ c_1x + c_0 \end{pmatrix} . e^{-2x} \Rightarrow \begin{aligned} y_1 &= (a_1x + a_0).e^{-2x} \\ y_2 &= (b_1x + b_0).e^{-2x} \\ y_3 &= (c_1x + c_0).e^{-2x} \end{aligned}$$

$y_1' = a_1 . e^{-2x} + (a_1 x + a_0). e^{-2x}. (-2) = -2. (a_1 x + a_0). e^{-2x} + 16. (b_1 x + b_0). e^{-2x} - 4. (c_1 x + c_0). e^{-2x}$

$y_2' = b_1 . e^{-2x} + (b_1 x + b_0). e^{-2x}. (-2) = 2. (b_1 x + b_0) - (c_1 x + c_0). e^{-2x}$ $\quad |: e^{-2x}$

$y_3' = c_1 . e^{-2x} + (c_1 x + c_0). e^{-2x}. (-2) = 2. (a_1 x + a_0) - 4. (c_1 x + c_0). e^{-2x}$

$a_1 = 16. (b_1 x + b_0) - 4. (c_1 x + c_0)$

$b_1 = 4. (b_1 x + b_0) - (c_1 x + c_0)$

$c_1 = 2. (a_1 x + a_0) - 2. (c_1 x + c_0)$

$0x + a_1 = (16b_0 - 4c_0) + (16b_1 - 4c_1). x$

$0x + b_1 = (4b_0 - c_0) + (4b_1 - c_1). x$

$0x + c_1 = (2a_0 - 2c_0) + (2a_1 - 2c_1). x$

$16b_1 - 4c_1 = 0 \qquad 16b_0 - 4c_0 = a_1$

$4b_1 - c_1 = 0 \qquad 4b_0 - c_0 = b_1$

$2a_1 - 2c_1 = 0 \qquad 2a_0 - 2c_0 = c_1$

Např. :$a_1 = 4, b_1 = 1, c_1 = 4, a_0 = 1, b_0 = 0, c_0 = -1$

$$^{3}\vec{y} = \begin{pmatrix} 4x+1 \\ x \\ 4x-1 \end{pmatrix}. e^{-2x}$$

$$\vec{y} = c_1.\ {}^{1}\vec{y} + c_2.\ {}^{2}\vec{y} + c_3.\ {}^{3}\vec{y} = c_1. \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + c_2. \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}. e^{-2x} + c_3. \begin{pmatrix} 4x+1 \\ x \\ 4x-1 \end{pmatrix}. e^{-2x}$$

3. Najděte všechna komplexní čísla z, pro která platí $\sinh z = \frac{e^z - e^{-z}}{2} = -2i$.

$$\frac{e^z - e^{-z}}{2} = -2i \quad |.2$$

$$e^z - \frac{1}{e^z} = -4i \quad |. e^z$$

$$(e^z)^2 + 4i. e^z - 1 = 0 \qquad \text{subst.}: e^z = a$$

$$a^2 + 4i. a - 1 = 0$$

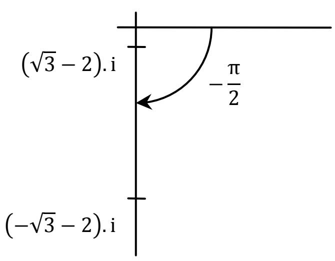
$$D = (4i)^2 - 4.1. (-1) = 16i^2 + 4 = -16 + 4 = -12 = 12i^2$$

$$a_{1,2} = \frac{-4i \pm i. \sqrt{12}}{2} = \frac{-4i \pm i. 2. \sqrt{3}}{2} \Rightarrow \begin{matrix} a_1 = -2i + i. \sqrt{3} = (\sqrt{3} - 2). i \\ a_2 = -2i - i. \sqrt{3} = (-\sqrt{3} - 2). i \end{matrix}$$

Pro $a_1 = (\sqrt{3} - 2). i$:

$$e^z = (\sqrt{3} - 2). i \Rightarrow z = \mathrm{Ln}[(\sqrt{3} - 2). i]$$

$$z = \ln|(\sqrt{3} - 2). i| + i. \left(-\frac{\pi}{2} + 2k\pi\right)$$



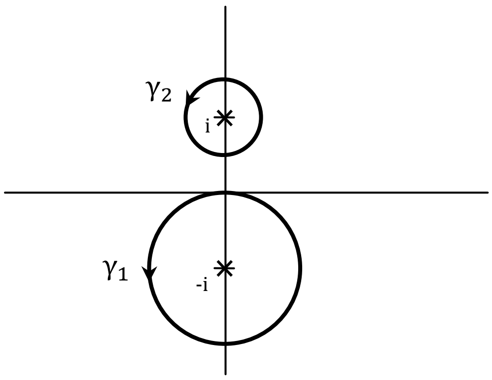
Pro $a_2 = (-\sqrt{3} - 2). i$:

$$e^z = (-\sqrt{3} - 2). i \Rightarrow z = \mathrm{Ln}[(-\sqrt{3} - 2). i]$$

$$z = \ln|(-\sqrt{3} - 2). i| + i. \left(-\frac{\pi}{2} + 2k\pi\right)$$

4. Vypočítejte $\int_\gamma \frac{z}{(z-i).(z+i)^2}.dz$, kde γ je kladně orientovaná kružnice.
   a) $\gamma_1: |z-i| = 1$
   b) $\gamma_2: |z+i| = \frac{1}{2}$ Nakreslete příslušný obrázek.

Stacionární body: $\pm i$



$$\gamma_1: f(z) = \frac{z}{(z+i)^2}, z_0 = i$$

$$\int_{\gamma_1} \frac{\frac{z}{(z+i)^2}}{z-i}.dz = 2\pi i. f(z_0) = 2\pi i.\frac{i}{(2i)^2} = \frac{2\pi i^2}{4i^2} = \frac{1}{2}\pi$$

Podle Cauchyova integračního vzorce.

$$\gamma_2: f(z) = \frac{z}{z-i}, z_0 = -i, n = 1$$

$$\int_{\gamma_2} \frac{\frac{z}{z-i}}{(z+i)^2}.dz = \frac{2\pi i}{1!}.f'(z_0) = 2\pi i.\frac{-i}{(-2i)^2} = \frac{-2\pi i^2}{4i^2} = -\frac{1}{2}\pi$$

$$f'(z) = \frac{1.(z-i) - z.1}{(z-i)^2} = \frac{-i}{(z-i)^2}$$

Podle Cauchyova integračního vzorce.

5. Pomocí Laplaceovy transformace najděte partikulární řešení diferenciální rovnice $y''' + 3y'' + 3y' + y = e^{-x}$, které splňuje počáteční podmínku $y(0) = 1, y'(0) = -1$ $y'' = 1$.

$$\mathcal{L}\{y\} = Y, \mathcal{L}\{y'\} = pY - 1, \mathcal{L}\{y''\} = p^2Y - p + 1, \mathcal{L}\{y'''\} = p^3Y - p^2 + p - 1, \mathcal{L}\{e^{-x}\} = \frac{1}{p+1}$$

$$p^3Y - p^2 + p - 1 + 3.(p^2Y - p + 1) + 3.(pY - 1) + Y = \frac{1}{p+1}$$

$$p^3Y - p^2 + p - 1 + 3p^2Y - 3p + 3 + 3pY - 3 + Y = \frac{1}{p+1}$$

$$Y.(p^3 + 3p^2 + 3p + 1) = \frac{1}{p+1} + p^2 - p + 1 + 3p = \frac{1}{p+1} + p^2 + 2p + 1 = \frac{1 + (p+1)^3}{p+1}$$

$$Y.(p+1)^3 = \frac{1 + (p+1)^3}{p+1}$$

$$Y = \frac{1 + (p+1)^3}{(p+1)^4} = \frac{1}{(p+1)^4} + \frac{(p+1)^3}{(p+1)^4} = \frac{1}{(p+1)^4} + \frac{1}{p+1}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(p+1)^4} + \frac{1}{p+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(p+1)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{p+1}\right\} = \frac{x^3 . e^{-x}}{6} + e^{-x}$$

$$\boxed{\frac{1}{(p-a)^{n+1}} \Rightarrow \frac{x^n . e^{ax}}{n!}}$$

6.

a. Zjistěte, pro která komplexní čísla z splňuje funkce $f(z) = \frac{|z|^2}{\bar{z}}$ Cauchy – Riemannovy podmínky.

b. Odvoďte vztah pro první aproximaci x_1 kořene α rovnice $f(x) = 0$ při použití metody regula – falsi (metoda sečen). Nultá aproximace je x_0, a je krajní bod separačního intervalu. Nakreslete.

c. Výpočtem pomocí definičního intervalu odvoďte obraz funkce $f(x) = 1$ v Laplaceově transformaci.