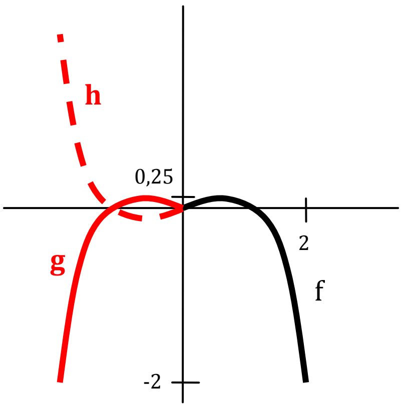
1. Je dána funkce $f(x) = x.(1 - x)$ v intervalu $(0; 2)$. Nakreslete graf sudého rozšíření **g** a lichého rozšíření **h** této funkce na interval $(-2; 2)$. Pak rozviňte funkci f intervalu $(0; 2)$ v kosinovou Fourierovu řadu .



$f(x) = x.(1 - x) = -x^2 + x$

Výpočet vrcholu paraboly:

$$x_0 = \frac{-b}{2a} = \frac{-1}{2.(-1)} = \frac{1}{2}$$

$$y_0 = f(x_0) = -\left(\frac{1}{2}\right)^2 + \frac{1}{2} = -\frac{1}{4} + \frac{1}{2} = -\frac{1}{4} + \frac{2}{4} = \frac{1}{4}$$

$2\ell = 4$

$\ell = 2$

$b_n = 0$

$$a_0 = \frac{1}{2}.2.\int_0^2 (-x^2 + x).dx = \left[-\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2 = -\frac{8}{3} + 2 = -\frac{8}{3} + \frac{6}{3} = -\frac{2}{3}$$

$u' = -2x + 1 \quad v = \frac{1}{n\frac{\pi}{2}}.\sin n\frac{\pi}{2}x = \frac{2}{n\pi}.\sin n\frac{\pi}{2}x$

$$a_n = \frac{1}{2}.2.\int_0^2 \overbrace{(-x^2 + x)}^{u}.\overbrace{\cos n\frac{\pi}{2}x}^{v'}.dx =$$

$u' = -2 \quad v = -\frac{2}{n\pi}.\cos n\frac{\pi}{2}x$

$$= \frac{2}{n\pi}.\underbrace{\left[(-x^2 + x).\sin n\frac{\pi}{2}x\right]_0^2}_{=0} - \frac{2}{n\pi}\int_0^2 \overbrace{(-2x + 1)}^{u}.\overbrace{\sin n\frac{\pi}{2}x}^{v'}.dx =$$

$$= -\frac{2}{n\pi}.\left\{-\frac{2}{n\pi}.\left[(-2x + 1).\cos n\frac{\pi}{2}x\right]_0^2 - 2.\frac{2}{n\pi}.\int_0^2 \cos n\frac{\pi}{2}x.dx\right\} =$$

$$= \frac{4}{n^2\pi^2}.\left\{-3.(-1)^n - 1 + 2.\frac{2}{n\pi}\underbrace{\left[\sin n\frac{\pi}{2}x\right]_0^2}_{=0}\right\} = \frac{4}{n^2\pi^2}.[-3.(-1)^n - 1]$$

$$x.(1 - x) \sim -\frac{1}{3} + \frac{4}{\pi^2}.\sum_{n=1}^{\infty}\frac{1}{n^2}.[-3.(-1)^n - 1].\cos n\frac{\pi}{2}x$$

2. Najděte obecné řešení soustavy diferenciálních rovnic:

$$y_1' = 2y_1 - y_2$$

$$y_2' = 5y_1 - 2y_2 + \frac{1}{\sin x}$$

Nepoužívejte eliminační metodu!

$$|A - \lambda E| = \begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = (2-\lambda).(-2-\lambda) + 5 = -4 - 2\lambda + 2\lambda + \lambda^2 + 5 = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1 = i^2 \Rightarrow \lambda_{1,2} = \sqrt{i^2} = \pm i$$

Pro $\lambda_1 = i$:

$$\begin{array}{r} (2-i).k_1 - k_2 = 0 \\ \underline{5k_1 + (-2-i).k_2 = 0} \end{array}$$

$$(2-2i).k_1 = k_2 \quad \text{např.}\ k_1 = 1, k_2 = 2 - i$$

$$\vec{y} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}.e^{ix} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i.\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right].(\cos x + i.\sin x) =$$

$$= \underbrace{\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}.\cos x - \begin{pmatrix} 0 \\ -1 \end{pmatrix}.\sin x\right]}_{{}^1\vec{y}} + i.\underbrace{\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}.\cos x + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.\sin x\right]}_{{}^2\vec{y}}$$

$$\vec{y}_h = c_1.\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}.\cos x - \begin{pmatrix} 0 \\ -1 \end{pmatrix}.\sin x\right] + c_2.\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}.\cos x + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.\sin x\right] \Rightarrow$$

$$\Rightarrow \begin{array}{l} y_1 = c_1.\cos x + c_2.\sin x \\ y_2 = c_1.(2\cos x + \sin x) + c_2.(-\cos x + 2\sin x) \end{array}$$

$$\begin{array}{r} c_1'(x).\cos x + c_2'(x).\sin x = 0 \\ \underline{c_1'(x).(2\cos x + \sin x) + c_2'(x).(-\cos x + 2\sin x) = \frac{1}{\sin x}} \end{array}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ 2\cos x + \sin x & -\cos x + 2\sin x \end{vmatrix} = -\cos^2 x + 2\sin x\cos x - 2\sin x\cos x - \sin^2 x =$$

$$= -\underbrace{(\sin^2 x + \cos^2 x)}_{1} = -1$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \frac{1}{\sin x} & -\cos x + 2\sin x \end{vmatrix}}{W} = \frac{-1}{-1} = 1 \Rightarrow c_1(x) = \int 1.dx = x$$

$$c_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ 2\cos x + \sin x & \frac{1}{\sin x} \end{vmatrix}}{W} = \frac{\frac{\cos x}{\sin x}}{-1} = -\frac{\cos x}{\sin x} \Rightarrow c_2(x) = \int -\frac{\cos x}{\sin x}.dx = \begin{vmatrix} \sin x = t \\ \cos x.dx = dt \end{vmatrix} =$$

$$= \int -\frac{\cos x}{t}.\frac{dt}{\cos x} = -\int \frac{1}{t}.dt = -\ln|t| = -\ln|\sin x|$$

$$\vec{y}_p: \begin{array}{l} y_1 = x.\cos x - \ln|\sin x|.\sin x \\ y_2 = x.(2\cos x + \sin x) - \ln|\sin x|.(-\cos x + 2\sin x) \end{array}$$

$$\vec{y} = \vec{y}_h + \vec{y}_p \Rightarrow$$

$$\Rightarrow \begin{array}{l} y_1 = c_1.\cos x + c_2.\sin x + x.\cos x - \ln|\sin x|.\sin x \\ y_2 = c_1.(2\cos x + \sin x) + c_2.(-\cos x + 2\sin x) + x.(2\cos x + \sin x) - \ln|\sin x|.(-\cos x + 2\sin x) \end{array}$$

3. Dokažte, že funkce $v(x; y) = e^x(\cos y + \sin y)$ je harmonická v R^2. Pak určete holomorfní funkci f, víte-li, že $\operatorname{Im} f = v(x; y)$ a $f(0) = i$. Zapište derivaci $f'(x + iy)$. (Nemusíte ji vyjadřovat jako funkci proměnné z).

$$v(x; y) = e^x(\cos y + \sin y) = e^x.\cos y + e^x.\sin y$$

$$\frac{\partial v}{\partial x} = e^x.\cos y + e^x.\sin y \qquad \frac{\partial^2 v}{\partial x^2} = e^x.\cos y + e^x.\sin y$$

$$\frac{\partial v}{\partial y} = -e^x.\sin y + e^x.\cos y \qquad \frac{\partial^2 v}{\partial y^2} = -e^x.\cos y - e^x.\sin y$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x.\cos y + e^x.\sin y - e^x.\cos y - e^x.\sin y = 0 - \text{funkce je harmonická}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -e^x.\sin y + e^x.\cos y \Rightarrow u(x; y) = \int (-e^x.\sin y + e^x.\cos y).dx =$$
$$= -e^x.\sin y + e^x.\cos y + \varphi(y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x.\cos y - e^x.\sin y + \varphi'(y) = -e^x.\cos y - e^x.\sin y \Rightarrow \varphi'(y) = 0 \Rightarrow \varphi(y) =$$
$$= \int 0.dy = 0 + K$$

$$u(x; y) = -e^x.\sin y + e^x.\cos y + K$$

$$f(x + iy) = -e^x.\sin y + e^x.\cos y + K + i.(e^x.\cos y + e^x.\sin y)$$

$$f(0) = f(0 + i0) = i = 1 + K + i.(1) \Rightarrow K = -1$$

$$f(x + iy) = -e^x.\sin y + e^x.\cos y - 1 + i.(e^x.\cos y + e^x.\sin y)$$

$$f'(x + iy) = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x} = -e^x.\sin y + e^x.\cos y + i.(e^x.\cos y + e^x.\sin y)$$

4. Vypočtěte:

$$\int_\gamma \frac{z}{(z^2 - 1).(z - 1)^2}.dz$$

, kde γ je kladně orientovaná kružnice:

a) $\gamma_1: \left|z + \frac{3}{2}\right| = 1$

b) $\gamma_2: \left|z - \frac{3}{2}\right| = 1$

c) $\gamma_3: |z| = 3$

Nakreslete a vše dostatečně zdůvodněte.

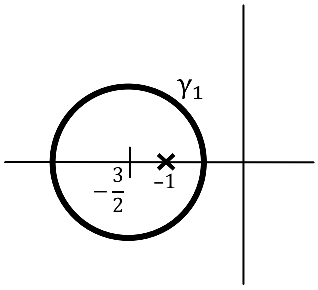
$$\int_\gamma \frac{z}{(z^2 - 1).(z - 1)^2}.dz = \int_\gamma \frac{z}{(z - 1).(z + 1).(z - 1)^2}.dz = \int_\gamma \frac{z}{(z + 1).(z - 1)^3}.dz$$

Stacionární body:

$z_0 = \pm 1$

a)

$\gamma_1: \left|z + \frac{3}{2}\right| = 1$



$z_0 = -1$

$$\int_{\gamma_1} \frac{\frac{z}{(z-1)^3}}{z+1} . dz$$

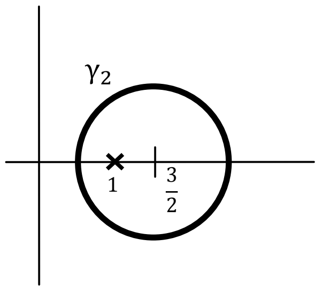
$$f(z) = \frac{z}{(z-1)^3}$$

$$\int_{\gamma_1} \frac{\frac{z}{(z-1)^3}}{z+1} . dz = 2\pi i . f(z_0) = 2\pi i . f(-1) = 2\pi i . \frac{-1}{-8} = \frac{1}{4}\pi i$$

Podle Cauchyova integračního vzorce.

b)

$\gamma_2: \left|z - \frac{3}{2}\right| = 1$



$z_0 = 1$

$$\int_{\gamma_2} \frac{\frac{z}{z+1}}{(z-1)^3} . dz$$

$$f(z) = \frac{z}{z+1}, n = 2$$

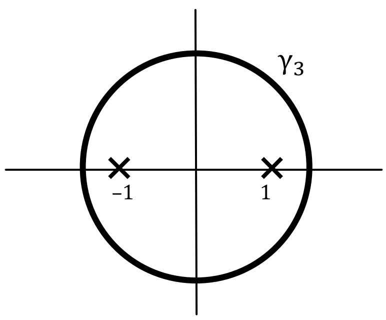
$$\int_{\gamma_2} \frac{\frac{z}{z+1}}{(z-1)^3} . dz = \frac{2\pi i}{2!} . f''(z_0) = \pi i . \frac{-2}{8} = -\frac{2}{8} . \pi i$$

$$f'(z) = \frac{1.(z+1) - z.1}{(z+1)^2} = \frac{1}{(z+1)^2} = (z+1)^{-2}, f''(z) = -2.(z+1)^{-3} = \frac{-2}{(z+1)^3}$$

Podle Cauchyova integračního vzorce.

c)

$\gamma_3: |z| = 3$



$$\int_{\gamma_3} \frac{z}{(z^2-1).(z-1)^2} . dz = \int_{\gamma_1} \quad + \int_{\gamma_2} \quad = \frac{1}{4}\pi i - \frac{2}{8}\pi i =$$

$$= \pi i . \left(\frac{2}{8} - \frac{2}{8}\right) = 0$$

Podle zobecněného principu deformace křívky.

5. Pomocí Laplaceovy transformace najděte partikulární řešení rovnice $y''' + y'' = e^{-x}$, které splňuje počáteční podmínku $y(0) = 0, y'(0) = 1, y''(0) = 0$.

$$y''' + y'' = e^{-x}$$

$$\mathcal{L}\{y\} = Y, \mathcal{L}\{y''\} = p^2Y - p.0 - 1, \mathcal{L}\{y'''\} = p^3Y - p^2.0 - p.1 - 0, \mathcal{L}\{e^{-x}\} = \frac{1}{p+1}$$

$$p^3Y - p + p^2Y - 1 = \frac{1}{p+1}$$

$$Y.(p^3 + p^2) = \frac{1}{p+1} + p + 1 = \frac{1 + (p+1)^2}{p+1} = \frac{p^2 + 2p + 2}{p+1}$$

$$Y = \frac{p^2 + 2p + 2}{(p+1).(p+1).p^2} = \frac{p^2 + 2p + 2}{(p+1)^2.p^2} = \frac{A}{p+1} + \frac{B}{(p+1)^2} + \frac{C}{p} + \frac{D}{p^2} =$$

$$= \frac{A.(p+1).p^2 + B.p^2 + C.(p+1)^2.p + D.(p+1)^2}{(p+1)^2.p^2} =$$

$$= \frac{A.(p^3 + p^2) + Bp^2 + Cp.(p^2 + 2p + 1) + D.(p^2 + 2p + 1)}{(p+1)^2.p^2} =$$

$$= \frac{A.(p^3 + p^2) + Bp^2 + C.(p^3 + 2p^2 + p) + D.(p^2 + 2p + 1)}{(p+1)^2.p^2}$$

$$\begin{aligned} A + C &= 0 \\ A + B + 2C + D &= 1 \\ C + 2D &= 2 \\ D &= 2 \end{aligned}$$

$A = 2, B = 1, C = -2, D = 2$

$$y = \mathcal{L}^{-1}\left\{\frac{2}{p+1} + \frac{1}{(p+1)^2} - \frac{2}{p} + \frac{2}{p^2}\right\} =$$

$$= 2.\mathcal{L}^{-1}\left\{\frac{1}{p+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(p+1)^2}\right\} - 2.\mathcal{L}^{-1}\left\{\frac{1}{p}\right\} + 2.\mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\} =$$

$$= 2.e^{-x} + x.e^{-x} - 2 + 2x$$

6.

a) Zapiště v algebraickém nebo goniometrickém tvaru všechny hodnoty $\sqrt{-2i}$.

$$\sqrt{-2i} = z \Rightarrow -2i = z^2$$

$$z = x + iy$$

$$-2i = (x + iy)^2 = x^2 - y^2 + i.2xy$$

$$\begin{aligned} x^2 - y^2 &= 0 \\ 2xy &= -2 \Rightarrow y = -\frac{1}{x} \end{aligned}$$

$$x^2 - \left(-\frac{1}{x}\right)^2 = 0$$

$x^2 - \frac{1}{x^2} = 0 \;\; |.\, x^2$

$x^4 - 1 = 0$

$x^2 = \pm 1$

$x^2 + 1 = 0 -$ nelze v R

$x^2 - 1 = 0$

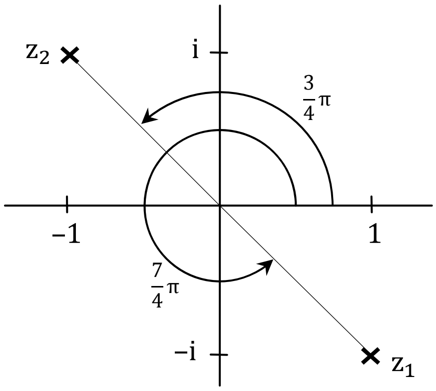
$(x + 1).(x - 1) = 0$

Pro $x = 1 \Rightarrow y = -1$

Pro $x = -1 \Rightarrow y = 1$

$z_1 = 1 - i$

$z_2 = -1 + i$



$z_1 = |1 - i|.\left(\cos\frac{7}{4}\pi + i.\sin\frac{7}{4}\pi\right) = \sqrt{2}.\left(\cos\frac{7}{4}\pi + i.\sin\frac{7}{4}\pi\right)$

$z_2 = |-1 + i|.\left(\cos\frac{3}{4}\pi + i.\sin\frac{3}{4}\pi\right) = \sqrt{2}.\left(\cos\frac{3}{4}\pi + i.\sin\frac{3}{4}\pi\right)$

b) Odvoďte vztah pro první aproximaci x_1 kořene α rovnice $f(x) = 0$ při použití metody regula falsi (metoda sečen). Nultá aproximace je x_0, a je krajní bod separačního intervalu. Nakrestele.

c) Zapište první tři členy Taylorova rozvoje se středem v bodě 1 řešení diferenciální rovnice $y' + 2xy - x = 0$, je-li $y(1) = -1$.