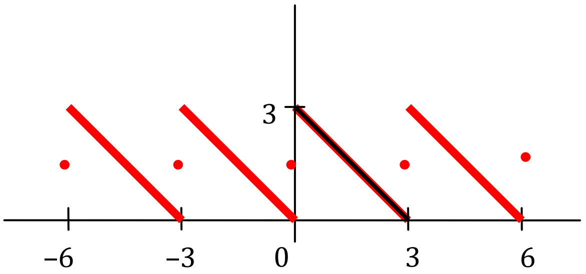
1. Rozviňte funkci $f(x) = 3 - x$ v intervalu $(0; 3)$ ve Fourierovu řadu. Nakreslete graf normalizovaného periodického pokračování $\bar{f}$ této funkce a určete $\bar{f}(-3)$.



$$2\ell = 3$$

$$\ell = \frac{3}{2}$$

$$\bar{f}(-3) = 3$$

$$a_0 = \frac{1}{\ell}.\int_0^3 (3-x).\,dx = \frac{2}{3}.\left[3x - \frac{x^2}{2}\right]_0^3 = \frac{2}{3}.\left(9 - \frac{9}{2}\right) = \frac{1}{3}.(18-9) = 3$$

$$u' = -1 \qquad v = \frac{3}{2\pi n}.\sin n\frac{2\pi}{3}x$$

$$a_n = \frac{2}{3}.\int_0^3 \overbrace{(3-x)}^{u}.\overbrace{\cos n\frac{\pi}{\frac{3}{2}}x}^{v'}.\,dx = \frac{2}{3}.\left\{\frac{3}{2\pi n}.\underbrace{\left[(3-x).\sin n\frac{2\pi}{3}x\right]_0^3}_{=0} + \frac{3}{2\pi n}.\int_0^3 \sin n\frac{2\pi}{3}x.\,dx\right\} =$$

$$= \frac{1}{n\pi}.\left(-\frac{3}{2\pi n}\right)\left[\cos n\frac{2\pi}{3}x\right]_0^3 = -\frac{3}{2n^2\pi^2}.[1-1] = 0$$

$$u' = -1 \qquad v = -\frac{3}{2\pi n}.\cos n\frac{2\pi}{3}x$$

$$b_n = \frac{2}{3}.\int_0^3 \overbrace{(3-x)}^{u}.\overbrace{\sin n\frac{2\pi}{3}x}^{v'}.\,dx = \frac{2}{3}.\left\{-\frac{3}{2\pi n}.\left[(3-x).\cos n\frac{2\pi}{3}x\right]_0^3 - \frac{3}{2\pi n}.\int_0^3 \cos n\frac{2\pi}{3}x.\,dx\right\} =$$

$$= -\frac{1}{\pi n}.\left\{-3 + \underbrace{\left[\sin n\frac{2\pi}{3}x\right]_0^3}_{=0}\right\} = \frac{3}{n\pi}$$

$$3 - x \sim \frac{3}{2} + \frac{3}{\pi}.\sum_{n=1}^{\infty}\frac{1}{n}.\sin n\frac{2\pi}{3}x$$

2. Určete obecné řešení soustavy diferenciálních rovnic

$$\begin{aligned}
y_1' &= -y_1 + 2y_2 + y_3\\
y_2' &= -2y_1 - y_2 + y_3\\
y_3' &= + 4y_2 + y_3
\end{aligned}$$

$$|A-\lambda E| = \begin{vmatrix} -1-\lambda & 2 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 4 & 1-\lambda \end{vmatrix} =$$

$$= (-1-\lambda).\begin{vmatrix} -1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} - 2.\begin{vmatrix} -2 & 1 \\ 0 & 1-\lambda \end{vmatrix} + 1.\begin{vmatrix} -2 & -1-\lambda \\ 0 & 4 \end{vmatrix} =$$

$$= (-1-\lambda).[(-1-\lambda).(1-\lambda)-4] - 2.(-2+2\lambda) + 1.(-8) =$$

$$= (-1-\lambda).(-1+\lambda-\lambda+\lambda^2-4) + 4 - 4\lambda - 8 = (-1-\lambda).(\lambda^2-5) - 4\lambda - 4 =$$

$$= (-1-\lambda).(\lambda^2-5) + 4.(-1-\lambda) = (-1-\lambda).(\lambda^2-1) = 0$$

$\lambda_1 = 1, \lambda_{2,3} = -1$

Pro $\lambda_1 = 1$:

$$\begin{array}{cccc} -2k_1 & +2k_2 & +1k_3 & =0 \\ -2k_1 & -2k_2 & +1k_3 & =0 \\ & +4k_2 & & =0 \end{array}$$

např. $k_2 = 0, k_3 = 2, k_1 = 1$

$${}^{1}\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.e^{x}$$

Pro $\lambda_2 = -1$:

$$\begin{array}{cccc} & +2k_2 & +1k_3 & =0 \\ -2k_1 & & +1k_3 & =0 \\ & +4k_2 & +2k_3 & =0 \end{array}$$

např. $k_1 = -1, k_2 = 1, k_3 = -2$

$${}^{2}\vec{y} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}.e^{-x}$$

Pro $\lambda_3 = -1$:

$${}^{3}\vec{y} = \begin{pmatrix} a_1x+a_0 \\ b_1x+b_0 \\ c_1x+c_0 \end{pmatrix}.e^{-x} \Rightarrow \begin{array}{l} y_1 = (a_1x+a_0).e^{-x} \\ y_2 = (b_1x+b_0).e^{-x} \\ y_3 = (c_1x+c_0).e^{-x} \end{array}$$

$$\begin{array}{l} y_1' = a_1.e^{-x} + (a_1x+a_0).(-1).e^{-x} = -(a_1x+a_0).e^{-x} + 2.(b_1x+b_0).e^{-x} + (c_1x+c_0).e^{-x} \\ y_2' = b_1.e^{-x} + (b_1x+b_0).(-1).e^{-x} = -2.(a_1x+a_0).e^{-x} - (b_1x+b_0).e^{-x} + (c_1x+c_0).e^{-x} \quad |.e^{x} \\ y_3' = c_1.e^{-x} + (c_1x+c_0).(-1).e^{-x} = 4.(b_1x+b_0).e^{-x} + (c_1x+c_0).e^{-x} \end{array}$$

$$\begin{array}{l} a_1 = 2.(b_1x+b_0) + (c_1x+c_0) \\ b_1 = -2(a_1x+a_0) + (c_1x+c_0) \\ c_1 = 4(b_1x+b_0) + 2.(c_1x+c_0) \end{array}$$

$$\begin{array}{l} 0.x + a_1 = (2b_0+c_0) + (2b_1+c_1).x \\ 0.x + b_1 = (-2a_0+c_0) + (-2a_1+c_1).x \\ 0.x + c_1 = (4b_0+2c_0) + (4b_1+2c_1).x \end{array}$$

$$\begin{array}{cccc} & +2b_0 & +c_0 & =a_1 \\ -2a_0 & & +c_0 & =b_1 \\ & +4b_0 & +2c_0 & =c_1 \end{array} \qquad \begin{array}{cccc} & +2b_1 & +c_1 & =0 \\ -2a_1 & & +c_1 & =0 \\ & +4b_1 & +2c_1 & =0 \end{array}$$

např. $a_1 = 2, \ b_1 = -2, c_1 = 4; a_0 = 1, b_0 = 1, c_0 = 0$

$$\vec{y} = c_1.\ {}^1\vec{y} + c_2.\ {}^2\vec{y} + c_3.\ {}^3\vec{y} = c_1.\begin{pmatrix}1\\0\\3\end{pmatrix}.e^x + c_2.\begin{pmatrix}-1\\1\\-2\end{pmatrix}.e^{-x} + c_3.\begin{pmatrix}2x+1\\-2x+1\\4x\end{pmatrix}.e^{-x}$$

$$\vec{y}: \begin{aligned} y_1 &= c_1.e^x - c_2.e^{-x} + c_3.(2x+1).e^{-x} \\ y_2 &= c_2.e^{-x} + c_3.(-2x+1).e^{-x} \\ y_3 &= 3c_1.e^x - 2c_2.e^{-x} + 4x.c_3.e^{-x} \end{aligned}$$

3. Dokažte, že funkce $v(x;y) = -x^2 + y^2 + 6xy + 2x$ je harmonická v R^2. Pak najděte holomorfní funkci f v C tak, že $\operatorname{Im} f = v(x;y)$ a $f(2) = 0$. Sestavte derivaci $f'(x+iy)$ a vyjádřete ji jako funkci proměnné $z = x + iy$.

$$\frac{\partial v}{\partial x} = -2x + 6y + 2 \qquad \frac{\partial^2 v}{\partial x^2} = -2$$

$$\frac{\partial v}{\partial y} = 2y + 6x \qquad \frac{\partial^2 v}{\partial y^2} = 2$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -2 + 2 = 0 - \text{funkce je haromonická}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2y + 6x \Rightarrow u(x;y) = \int (2y+6x).dx = 2xy + 3x^2 + \varphi(y)$$

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -(-2x + 6y + 2) = 2x - 6y - 2 = 2x + \varphi'(y) \Rightarrow \varphi'(y) = -6y - 2$$

$$\varphi(y) = \int (-6y - 2).dy = -3y^2 - 2y + K$$

$$f(x+iy) = 2xy + 3x^2 - 3y^2 - 2y + K + i.(-x^2 + y^2 + 6xy + 2x)$$

$$f(2) = f(2 + i0) = 0 \Rightarrow f(2 + i0) = 12 + K + i.(-4 + 4) = 12 + K = 0$$

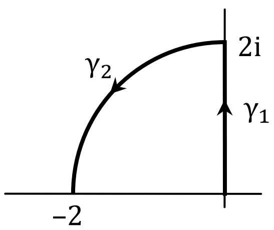
$$K = -12$$

$$u(x;y) = 2xy + 3x^2 - 3y^2 - 2y - 12$$

$$f(x+iy) = 2xy + 3x^2 - 3y^2 - 2y - 12 + i.(-x^2 + y^2 + 6xy + 2x)$$

$$f'(x+iy) = \frac{\partial u}{\partial x} + i.\frac{\partial v}{\partial x} = 2y + 6x + i.(-2x + 6y + 2)$$

4. Vypočítejte $\int_\gamma \frac{z}{\bar{z}}.dz$, kde γ je orientovaná křivka z obrázku.



γ_1:

$z = it, t \epsilon \langle 0,2 \rangle, dz = i.dt, \bar{z} = -it$

γ_2:

$z = e^{it}, t \epsilon \langle \frac{\pi}{2}; \pi \rangle, dz = i.e^{it}, \bar{z} = e^{-it}$

$$\int_{\gamma} \frac{z}{\bar{z}}.\,dz = \int_{\gamma_1} + \int_{\gamma_2} = -2i - \frac{1}{3} + i.\frac{1}{3} = -\frac{1}{3} + i.\left(\frac{1}{3} - 2\right) = -\frac{1}{3} + i.\left(-\frac{5}{3}\right)$$

$$\int_{\gamma_1} \frac{z}{\bar{z}}.\,dz = \int_0^2 \frac{it}{-it}.\,i.\,dt = -i.\int_0^2 dt = -i.\,[t]_0^2 = -i.\,2$$

$$\int_{\gamma_2} \frac{z}{\bar{z}}.\,dz = \int_{\frac{\pi}{2}}^{\pi} \frac{e^{it}}{e^{-it}}.\,i.\,e^{it}.\,dt = i.\int_{\frac{\pi}{2}}^{\pi} e^{3it}.\,dt = i.\left[\frac{e^{3it}}{3i}\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{3}.\left[e^{3\pi i} - e^{3\frac{\pi}{2}i}\right] =$$

$$= \frac{1}{3}.\left[\underbrace{\cos 3\pi}_{-1} + i.\underbrace{\sin 3\pi}_{0} - \underbrace{\cos\frac{3}{2}\pi}_{0} - i.\underbrace{\sin\frac{3}{2}\pi}_{-1}\right] = \frac{1}{3}.(-1+i) = -\frac{1}{3} + i.\frac{1}{3}$$

5. Pomocí Laplaceovy transformace najděte partikulární řešení integrodiferenciální rovnice $y' + 4.\int_0^x y(u).\,du = 3.\,e^{2x} + 2$ splňující počáteční podmínku $y(0) = -2$.

$$y' + 4.\int_0^x y(u).\,du = 3.\,e^{2x} + 2$$

$$\mathcal{L}\{y\} = Y, \mathcal{L}\{y'\} = pY + 2, \mathcal{L}\left\{\int_0^x y(u).\,du\right\} = \frac{1}{p}.\,Y, \mathcal{L}\{e^{2x}\} = \frac{1}{p-2}, \mathcal{L}\{1\} = \frac{1}{p}$$

$$pY + 2 + \frac{4}{p}Y = \frac{3}{p-2} + \frac{2}{p}$$

$$Y.\left(p + \frac{4}{p}\right) = \frac{3}{p-2} + \frac{2}{p} - 2 \quad |.\,p$$

$$Y.\,(p^2+4) = \frac{3p}{p-2} + 2 - 2p = \frac{3p + 2.\,(p-2) - 2p(p-2)}{p-2} = \frac{3p + 2p - 4 - 2p^2 + 4p}{p-2} =$$

$$= \frac{-2p^2 + 9p - 4}{p-2}$$

$$Y.\,(p^2+4) = \frac{-2p^2 + 9p - 4}{(p-2)}$$

$$Y = \frac{-2p^2 + 9p - 4}{(p^2+4).\,(p-2)} = \frac{Ap + B}{p^2+4} + \frac{C}{p-2} = \frac{(Ap+B).\,(p-2) + C.\,(p^2+4)}{(p^2+4).\,(p-2)} =$$

$$= \frac{Ap^2 - 2Ap + Bp - 2B + Cp^2 + 4C}{(p^2+4).\,(p-2)}$$

$$\begin{array}{llll} A & & +C & = -2 \\ -2A & +B & & = 9 \\ & -2B & +4C & = -4 \\ \hline \end{array}$$

$$A = -\frac{11}{4}, B = \frac{7}{2} = 2.\frac{7}{4}, C = \frac{3}{4}$$

$$Y = \frac{-\frac{11}{3}p + 2.\frac{7}{4}}{p^2 + 4} + \frac{\frac{3}{4}}{p - 2} = -\frac{11}{3}.\frac{p}{p^2 + 4} + \frac{7}{4}.\frac{2}{p^2 + 4} + \frac{3}{4}.\frac{1}{p - 2}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{-\frac{11}{3}.\frac{p}{p^2 + 4} + \frac{7}{4}.\frac{2}{p^2 + 4} + \frac{3}{4}.\frac{1}{p - 2}\right\} =$$

$$= -\frac{11}{3}.\mathcal{L}^{-1}\left\{\frac{p}{p^2 + 4}\right\} + \frac{7}{4}.\mathcal{L}^{-1}\left\{\frac{2}{p^2 + 4}\right\} + \frac{3}{4}.\mathcal{L}^{-1}\left\{\frac{1}{p - 2}\right\} =$$

$$= -\frac{11}{3}.\cos 2x + \frac{7}{4}.\sin 2x + \frac{3}{4}.e^{2x}$$

6.

a) Výpočtem dokažte, že funkce $f_1(x) = \cos x$, $f_2(x) = \sin x$ splňují na intervalu $\langle -\pi; \pi \rangle$ podmínku ortogonality.

b) Rozhodněte, který z intervalů $\langle -1; 0 \rangle$, $\langle 0; 1 \rangle$ je separační interval kořene rovnice $x^3 + 3x - 2 = 0$. Zdůvodněte.

c) Odvoďte vztah pro aproximaci y_1 řešení diferenciální rovnice $y' = f(x; y), y(x_0) = y_0$, v bodě $x_1 = x_0 + h$ při použití modifikované Eulerovy metody. Nakreslete.