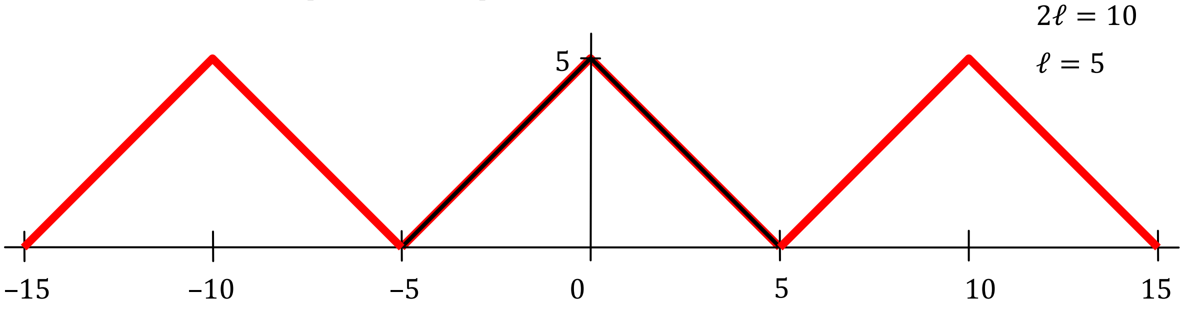
1. Rozviňte funkci $f(x) = 5 - |x|$ v intervalu $\langle -5; 5\rangle$ ve Fourierovu řadu. Nakreslete graf normalizovaného periodického pokračování $\bar{f}$ funkce f určete $\bar{f}(15)$.

$2\ell = 10$

$\ell = 5$



$\bar{f}(15) = 0$

v $\langle -5; 0\rangle$ $f(x) = 5 + x$

v $\langle 0; 5\rangle$ $f(x) = 5 - x$

Funkce je sudá $\Rightarrow$ kosinusová řada $b_n = 0$

$$a_0 = \frac{1}{5}.2.\int_0^5 (5-x).dx = \frac{2}{5}.\left[5x - \frac{x^2}{2}\right]_0^5 = \frac{2}{5}.\left[25 - \frac{25}{2} - 0\right] = \frac{1}{5}.(50-25) = 10 - 5 = 5$$

$$u' = -1 \quad v = \frac{5}{n\pi}.\sin n.\frac{\pi}{5}.x$$

$$a_n = \frac{1}{5}.2.\int_0^5 \overbrace{(5-x)}^{u}.\overbrace{\cos n.\frac{\pi}{5}.x}^{v'}.dx = \frac{2}{5}\left\{\frac{5}{n\pi}.\underbrace{\left[(5-x).\sin n.\frac{\pi}{5}.x\right]_0^5}_{=0} + \frac{5}{n\pi}.\int_0^5 \sin n.\frac{\pi}{5}.x.dx\right\} =$$

$$= \frac{2}{n\pi}.\left(-\frac{5}{n\pi}\right).\left[\cos n.\frac{\pi}{5}.x\right]_0^5 = -\frac{10}{n^2\pi^2}.[(-1)^n - 1] \quad \text{n liché } \frac{20}{n^2\pi^2}, \quad \text{n sudé } 0$$

$$n = 2k-1, a_{2k-1} = \frac{20}{(2k-1)^2\pi^2} \Rightarrow a_n = \frac{20}{(2n-1)^2\pi^2}$$

$$5 - |x| \sim \frac{5}{2} + \frac{20}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1).\frac{\pi}{5}.x$$

2. Najděte obecné řešení soustavy diferenciálních rovnic

$y_1' = -3y_1 + y_2$

$y_2' = -y_1 - y_2 + 16e^{2x}$

$$|A - \lambda E| = \begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (-3-\lambda).(-1-\lambda) + 1 = 3 + 3\lambda + \lambda + \lambda^2 + 1 = \lambda^2 + 4\lambda + 4 = 0$$

$D = 16 - 4.1.4 = 0$

$$\lambda_{1,2} = \frac{-4}{2} = -2$$

Pro $\lambda_1 = -2$:

$$-1k_1 + 1k_2 = 0$$
$$-1k_1 + 1k_2 = 0$$

$k_1 = k_2 = 1$

$$^1\vec{y} = \begin{pmatrix}1\\1\end{pmatrix}.e^{-2x}$$

Pro $\lambda_2 = -2$:

$$^2\vec{y} = \begin{pmatrix}a_1x + a_0\\b_1x + b_0\end{pmatrix}.e^{-2x} \quad \begin{matrix}y_1 = (a_1x + a_0).e^{-2x}\\y_2 = (b_1x + b_0).e^{-2x}\end{matrix}$$

$$y_1' = a_1.e^{-2x} + (a_1x + a_0).(-2).e^{-2x} = -3.(a_1x + a_0).e^{-2x} + (b_1x + b_0).e^{-2x}$$
$$y_2' = b_1.e^{-2x} + (b_1x + b_0).(-2).e^{-2x} = -(a_1x + a_0).e^{-2x} - (b_1x + b_0).e^{-2x} \quad |.e^{2x}$$

$$a_1 = -(a_1x + a_0) + (b_1x + b_0)$$
$$b_1 = -(a_1x + a_0) + (b_1x + b_0)$$

$$0.x + a_1 = (-a_0 + b_0) + (-a_1 + b_1)x$$
$$0.x + b_1 = (-a_0 + b_0) + (-a_1 + b_1)x$$

$$a_1 = -a_0 + b_0 \qquad -a_1 + b_1 = 0$$
$$b_1 = -a_0 + b_0 \qquad -a_1 + b_1 = 0$$

$a_1 = b_1 = 1, a_0 = 0, b_0 = 1$

$$^2\vec{y} = \begin{pmatrix}x\\x+1\end{pmatrix}.e^{-2x}$$

$$\vec{y}_h = c_1.\ ^1\vec{y} + c_2.\ ^2\vec{y} = c_1.\begin{pmatrix}1\\1\end{pmatrix}.e^{-2x} + c_2.\begin{pmatrix}x\\x+1\end{pmatrix}.e^{-2x}: \begin{matrix}y_1 = c_1.e^{-2x} + c_2.x.e^{-2x}\\y_2 = c_1.e^{-2x} + c_2.(x+1).e^{-2x}\end{matrix}$$

$$c_1'(x).e^{-2x} + c_2'(x).x.e^{-2x} = 0$$
$$c_1'(x).e^{-2x} + c_2'(x).(x+1).e^{-2x} = 16e^{2x} \quad |.e^{2x}$$

$$c_1'(x) + c_2'(x).x = 0$$
$$c_1'(x) + c_2'(x).(x+1) = 16e^{4x} \quad |-$$

$$c_2'(x).x - c_2'(x).x - c_2'(x) = -16e^{4x}$$

$$-c_2'(x) = -16e^{4x} \Rightarrow c_2'(x) = 16e^{4x}$$

$$c_2 = \int 16e^{4x}.dx = 16.\left[\frac{e^{4x}}{4}\right] = 4.e^{4x}$$

$$c_1'(x) + 16e^{4x}.x. = 0$$

$$c_1'(x) = -16e^{4x}.x$$

$$u' = 1 \quad v = \frac{1}{4}.e^{4x}$$

$$c_1(x) = \int -16.\overset{u}{\widetilde{x}}.\overset{v'}{\overbrace{e^{4x}}}.dx = -16.\left\{\frac{1}{4}.x.e^{4x} - \frac{1}{4}\int e^{4x}.dx\right\} = -4.\left\{x.e^{4x} - \frac{1}{4}e^{4x}\right\} =$$
$$= -4.x.e^{4x} + e^{4x} = e^{4x}.(-4x + 1)$$

$$\vec{y}_p: \begin{matrix}y_1 = e^{4x}(-4x+1).e^{-2x} + 4.e^{4x}.x.e^{-2x} = (-4x+1).e^{2x} + 4.e^{2x}.x\\y_2 = e^{4x}(-4x+1).e^{-2x} + 4.e^{4x}.(x+1).e^{-2x} = (-4x+1).e^{2x} + 4.e^{2x}.(x+1)\end{matrix}$$

$\vec{y} = \vec{y}_h + \vec{y}_p$

$\vec{y}: \begin{matrix} y_1 = c_1 e^{-2x} + c_2.x.e^{-2x} + (-4x+1).e^{2x} + 4.e^{2x}.x \\ y_2 = c_1.e^{-2x} + c_2.(x+1)e^{-2x} + (-4x+1).e^{2x} + 4.e^{2x}.(x+1) \end{matrix}$

3. Najděte všechna komplexní čísla z, pro která je $\operatorname{cotg} z = \frac{\cos z}{\sin z} = -2i$.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\frac{\frac{e^{iz} + e^{-iz}}{2}}{\frac{e^{iz} - e^{-iz}}{2i}} = -2i$$

$$\frac{2i.\left(e^{iz} + e^{-iz}\right)}{2.(e^{iz} - e^{-iz})} = -2i \quad |:i$$

$$\frac{e^{iz} + \frac{1}{e^{iz}}}{e^{iz} - \frac{1}{e^{iz}}} = -2 \quad \text{subst. } e^{iz} = a$$

$$\frac{a + \frac{1}{a}}{a - \frac{1}{a}} = \frac{\frac{a^2+1}{a}}{\frac{a^2-1}{a}} = \frac{a^2+1}{a^2-1} = -2$$

$$a^2 + 1 = -2a^2 + 2$$

$$3a^2 = 1 \Rightarrow a^2 = \frac{1}{3} \Rightarrow a = \pm\sqrt{\frac{1}{3}}$$

Pro $a_1 = \sqrt{\frac{1}{3}}$:

$$e^{iz} = \sqrt{\frac{1}{3}} \Rightarrow iz = \operatorname{Ln}\left(\sqrt{\frac{1}{3}}\right) = \ln\left|\sqrt{\frac{1}{3}}\right| + i.(0 + 2k\pi)$$

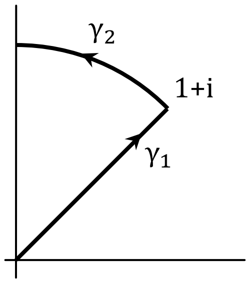
$$z = 2k\pi - i.\ln\left(\sqrt{\frac{1}{3}}\right) = 2k\pi - i.\frac{1}{2}.\ln\frac{1}{3}$$

Pro $a_2 = -\sqrt{\frac{1}{3}}$:

$$e^{iz} = -\sqrt{\frac{1}{3}} \Rightarrow iz = \operatorname{Ln}\left(-\sqrt{\frac{1}{3}}\right) = \ln\left|-\sqrt{\frac{1}{3}}\right| + i.(\pi + 2k\pi)$$

$$z = \pi + 2k\pi - i.\ln\left(\sqrt{\frac{1}{3}}\right) = \pi + 2k\pi - i.\frac{1}{2}.\ln\frac{1}{3}$$

4. Vypočítejte $\int_\gamma z.|z|.dz$, kde γ je orientovaná křivka z obrázku.



γ_1:

$z = t + it, t\epsilon\langle 0;1\rangle$

$dz = (1+i)dt, |z| = \sqrt{t^2+t^2} = \sqrt{2}.t$

γ_2:

$z = \sqrt{2}.e^{it}, t\epsilon\langle\frac{\pi}{4};\frac{\pi}{2}\rangle$

$dz = \sqrt{2}.i.e^{it}.dt, |z| = \sqrt{2}$

$$\int_\gamma z.|z|.dz = \int_{\gamma_1} + \int_{\gamma_2} = \frac{2}{3}.\sqrt{2}.i - \sqrt{2} - i.\sqrt{2}$$

$$\int_{\gamma_1} z.|z|.dz = \int_0^1 (t+it).\sqrt{2}.t.(1+i).dt = \sqrt{2}.\int_0^1 t.(t+it+it+i^2t).dt = \sqrt{2}.\int_0^1 2it^2.dt =$$

$$= \sqrt{2}.2.i.\left[\frac{t^3}{3}\right]_0^1 = \frac{2}{3}.\sqrt{2}.i$$

$$\int_{\gamma_2} z.|z|.dz = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}.e^{it}.\sqrt{2}.\sqrt{2}.i.e^{it}.dt = 2.\sqrt{2}.i.\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{2it}.dt = 2.\sqrt{2}.i.\left[\frac{e^{2it}}{2i}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sqrt{2}.\left(e^{\pi i} - e^{\frac{\pi}{2}i}\right) =$$

$$= \sqrt{2}.\left(\underbrace{\cos\pi}_{-1} + \underbrace{i.\sin\pi}_{0} - \underbrace{\cos\frac{\pi}{2}}_{0} - i.\underbrace{\sin\frac{\pi}{2}}_{1}\right) = \sqrt{2}.(-1-i) = -\sqrt{2} - i.\sqrt{2}$$

5. Pomocí Laplaceovy transformace najděte partikulární řešení soustavy diferenciálních rovnic

$y_1'' - y_2' = 0$

$y_2' - y_1' = 4e^x$

které splňují počáteční podmínku $y_1(0) = y_2'(0) = 0, y_1'(0) = y_2(0) = 2$

$\mathcal{L}\{y_1\} = Y_1, \mathcal{L}\{y_1'\} = pY_1 - 0, \mathcal{L}\{y_1''\} = p^2Y_1 - p.0 - 2$

$\mathcal{L}\{y_2\} = Y_2, \mathcal{L}\{y_2'\} = pY_2 - 2, \mathcal{L}\{y_2''\} = p^2Y_2 - p.2 - 0$

$$\mathcal{L}\{e^x\} = \frac{1}{p-1}$$

$p^2Y_1 - 2 - pY_2 + 2 = 0$

$$pY_2 - 2 - pY_1 = \frac{4}{p-1}$$

$p^2Y_1 - pY_2 = 0$

$$-pY_1 + pY_2 = \frac{4}{p-1} + 2 = \frac{4+2p-2}{p-1} = \frac{2p+2}{p-1} \quad |+$$

$$p^2Y_1 - pY_1 = \frac{2p+2}{p-1}$$

$$Y_1(p^2 - p) = \frac{2p+2}{p-1}$$

$$Y_1 = \frac{2p+2}{(p^2-p)(p-1)} = \frac{2p+2}{p.(p-1).(p-1)} = \frac{2p+2}{p.(p-1)^2} = \frac{A}{p} + \frac{B}{p-1} + \frac{C}{(p-1)^2} =$$

$$= \frac{A.(p-1)^2 + B.p.(p-1) + C.p}{p.(p-1)^2} = \frac{A.(p^2-2p+1) + B.p.(p-1) + C.p}{p.(p-1)^2}$$

$$A + B = 0$$
$$-2A - B + C = 2$$
$$A = 2$$

$$A = 2, B = -2, C = 4$$

$$p^2Y_1 - pY_2 = 0 \Rightarrow p^2.\frac{2p+2}{p.(p-1)^2} = pY_2$$

$$p.\frac{2p+2}{(p-1)^2} = Y_2 \Rightarrow Y_2 = \frac{2p+2}{(p-1)^2} = \frac{A}{p-1} + \frac{B}{(p-1)^2} = \frac{A.(p-1)+B}{(p-1)^2}$$

$$A = 2$$
$$-A + B = 2$$

$$A = 2, B = 4$$

$$y_1 = \mathcal{L}^{-1}\left\{\frac{2}{p} + \frac{-2}{p-1} + \frac{4}{(p-1)^2}\right\} = 2.\mathcal{L}^{-1}\left\{\frac{1}{p}\right\} - 2.\mathcal{L}^{-1}\left\{\frac{1}{p-1}\right\} + 4.\mathcal{L}^{-1}\left\{\frac{1}{(p-1)^2}\right\} =$$
$$= 2.1 - 2.e^x + 4.x.e^x$$

$$y_2 = \mathcal{L}^{-1}\left\{\frac{2}{p-1} + \frac{4}{(p-1)^2}\right\} = 2.\mathcal{L}^{-1}\left\{\frac{1}{p-1}\right\} + 4.\mathcal{L}^{-1}\left\{\frac{1}{(p-1)^2}\right\} = 2.e^x + 4.x.e^x$$

6.

a) Zapište reálnou a imaginární část funkce $f(z) = (\overline{z})^2$ a najděte ta komplexní čísla $z = x + iy$, pro které jsou splněny Cauchy-Reamanovy podmínky.

$$z = x + iy$$

$$f(x+iy) = \left(\overline{x+iy}\right)^2 = (x-iy)^2 = x^2 - 2ixy + i^2y^2 = (x^2 - y^2) + i.(-2xy)$$

$$\text{Re}\,[f(x+iy)] = x^2 - y^2 = u(x;y)$$

$$\text{Im}\,[f(x+iy)] = -2xy = v(x;y)$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = -2x \Leftrightarrow x = 0$$

$$\frac{\delta u}{\delta y} = -2y = -\frac{\partial v}{\partial x} = -(-2y) = 2y \Leftrightarrow y = 0$$

Cauchy-Riemanovy podmínky jsou splněny pro $x = 0$ a $y = 0$.

b) Zapište vztah pro n-tou aproximaci x_n kořene α při řešení rovnice $x = \varphi(x)$ obecnou iterační metodou. Nakreslete obrázek pro geometrickou interpretaci tohoto vztahu při podmínce $0 < \varphi'(x) < 1$ platné v separačním intervalu.

c) Pomocí definičního integrálu odvoďte obraz funkce $f(x) = x$ v Laplaceově transformaci pro $p > 1$.

$$\mathcal{L}\{x\} = \int_0^{+\infty} \overset{u}{\widetilde{x}} . \overbrace{e^{-px}}^{v'} . dx = -\frac{1}{p}[x.e^{-px}]_0^{+\infty} + \frac{1}{p} . \underbrace{\int_0^{+\infty} 1.e^{-px}.dx}_{\mathcal{L}\{1\}=\frac{1}{p}} = -\frac{1}{p} . \underbrace{\left[\lim_{x\to+\infty} \frac{x}{e^{px}} - 0\right]}_{\to 0} + \frac{1}{p^2} = \frac{1}{p^2}$$

$$u' = 1 \quad v = -\frac{1}{p} . e^{-px}$$