Poznámka: Systém funkcí $\{1; \cos nx; \sin nx\}$ je ortogonální na každém intervalu délky 2π, např. $\langle -\pi; \pi\rangle$.

Definice 2.2.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

Trigonometrická řada

$\left.\begin{matrix} a_n, n = 0,1,2,\dots \\ b_n, n = 1,2,\dots \end{matrix}\right\}$ reálná čísla (koeficienty řady)

a_0 – absolutní člen – konst.

$\forall n; a_n = 0$ – sinová trigonometrická řada
$\forall n; b_n = 0$ – kosinová trigonometrická řada

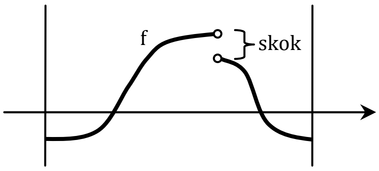
$$\int_{-\pi}^{\pi} 1.\cos nx\,dx = \int_{-\pi}^{\pi} 1.\sin nx\,dx = 0$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 0, m \neq n$$

$$\int_{-\pi}^{\pi} 1.1dx = 2\pi, \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi} \sin nx \sin nx\,dx = \pi$$

Odvození koeficientů Fourierovy trigonometrické řady

Uvažujme funkci $f(x)$ po částech spojitou v $\langle -\pi; \pi\rangle$, tzn. Má konečně mnoho bodů nespojitosti 1. druhu (konečný skok).



$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} f(x) = \int_{-\pi}^{\pi} \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,dx$$

$$\int_{-\pi}^{\pi} f(x) = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{n=1}^{\infty} \underbrace{\cos nx\,dx}_{=0} + b_n \int_{n=1}^{\infty} \underbrace{\sin nx\,dx}_{=0}\right)$$

$$\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}a_0 2\pi + 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad / \cos mx$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \underbrace{\cos mx\,dx}_{=0} + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \underbrace{\cos nx \cos mx\,dx}_{=0 \text{ jen pro } m\neq n} + b_n \int_{-\pi}^{\pi} \underbrace{\sin nx \cos mx}_{=0}\,dx\right)$$

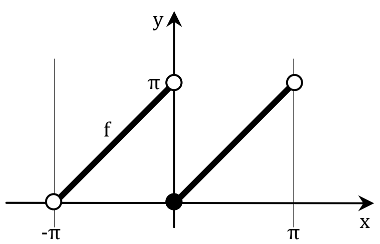
$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + a_m \int_{-\pi}^{\pi} \cos mx \cos mx\,dx = 0 + a_m\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

Stejně odvodíme $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Př. Sestavte Fourierovu řadu funkce $f(x) =$

$$= \begin{matrix} x+\pi, x\epsilon(-\pi;0) \\ x, x\epsilon\langle 0;\pi) \end{matrix} \text{ (Nezáleží na uzavřenosti/otevřenosti intervalu)}$$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x+\pi)dx + \int_{0}^{\pi} xdx\right\} =$$

$$= \frac{1}{\pi}\left\{\left[\frac{(x+\pi)^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right\} =$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2} - 0 + \frac{\pi^2}{2} - 0\right\} = \pi$$

$$u' = 1, v = \frac{\sin nx}{n}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(x+\pi)\cos nx\,dx + \int_{0}^{\pi} \overbrace{x}^{u}.\overbrace{\cos nx}^{v'}\,dx\right\} =$$

$$= \begin{vmatrix} u = x+\pi & u' = 1 \\ u' = \cos nx & v = \dfrac{\sin nx}{n} \end{vmatrix} =$$

$$= \frac{1}{\pi}\left\{\underbrace{\left[(x+\pi).\frac{\sin nx}{n}\right]_{-\pi}^{0}}_{=0} - \int_{-\pi}^{0}\frac{\sin nx}{n}dx + \underbrace{\left[x.\frac{\sin nx}{n}\right]_{0}^{\pi}}_{=0} - \frac{1}{n}\int_{0}^{\pi}\sin nx\,dx\right\} =$$

$$= \frac{1}{\pi n}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{\cos nx}{n}\right]_{0}^{\pi}\right\} = \frac{1}{\pi n}\left\{\underbrace{\cos 0}_{1} - \underbrace{\cos(-n\pi)}_{(-1)^n} + \underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\cos 0}_{1}\right\} = 0$$

$$u' = 1, v = \frac{-\cos nx}{n} \qquad u' = 1, v = \frac{-\cos nx}{n}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \overbrace{(x+\pi)}^{u} \overbrace{\sin nx}^{v'} \, dx + \int_{0}^{\pi} \overbrace{x}^{u} . \overbrace{\sin nx}^{v'} \, dx\right\} =$$

$$= \frac{1}{\pi}\left\{\left[(x+\pi).\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0} \cos nx \, dx + \left[x.\frac{-\cos nx}{n}\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi} \cos nx \, dx\right\} =$$

$$= \frac{1}{\pi n}\left\{-\pi - 0 + \underbrace{\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}}_{=0} - \pi.(-1)^n + 0 + \underbrace{\left[\frac{\sin nx}{n}\right]_{0}^{\pi}}_{=0}\right\} = -\frac{1}{n}\{1 + (-1)^n\} =$$

$$= \begin{cases} -\ 0, n \text{ liché} \\ -\ -\frac{2}{n}, n \text{ sudé } n = 2k \end{cases}$$

$$n = 2k \Rightarrow b_{2k} = -\frac{2}{2k} = -\frac{1}{k}$$

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)\{1 + (-1)^n\} \sin nx \qquad v\ (-\pi; \pi)$$

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{k}\right) \sin 2kx$$

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right) \sin 2nx = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin 2nx}{n} = \frac{\pi}{2} - \frac{\sin 2x}{1} - \frac{\sin 4x}{2} - \frac{\sin 6x}{3} - \cdots$$

Věta 2.3 + Věta 2.4

Nechť f je po částech spojitá funkce v $\langle -\pi; \pi \rangle$. Dále buď:

a) f' je po částech spojitá funkce v $\langle -\pi; \pi \rangle$, nebo
b) f má v $\langle -\pi; \pi \rangle$ konečně mnoho ostrých lokálních extrémů

Pak Fourierova řada funkce f konverguje k

- hodnotě $f(x)$ v bodech $x \epsilon (-\pi; \pi)$, kde je funkce f spojitá
- hodnotě $\underbrace{\frac{1}{2}\left[\lim_{x \to x_k^+} f(x) + \lim_{x \to x_k^-} f(x)\right]}_{\frac{1}{2}[f(x_k^+) + f(x_k^-)]}$ v bodech x_k nespojitosti funkce f
- hodnotě $\frac{1}{2}[\lim_{x \to -\pi^+} f(x) + \lim_{x \to \pi^-} f(x)]$ v bodech $\pm\pi$

$$\cancel{f(x)} \sim \underbrace{\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}_{\text{součet musí být periodická funkce v R}}$$

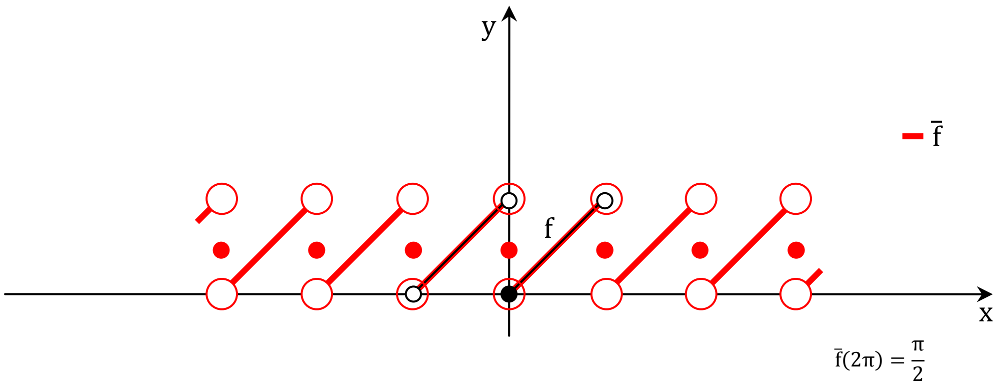
$\bar{f}(x)$ $\quad\bar{f}$ … normalizované periodické prodloužení

Definice 2.4. Normalizovaným periodickým pokročováním $\bar{f}$ funkce f nazýváme funkci definovanou v (-∞;+∞) takto:

$$\bar{f}(x) = \begin{cases} \frac{1}{2}\left[\lim_{t\to-\pi^+} f(t) + \lim_{t\to\pi^-} f(t)\right] \text{ pro } x = -\pi \\ \frac{1}{2}\left[\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)\right] \text{ pro } x\epsilon(-\pi;\pi) \text{ v bodech nespojitosti } x\epsilon(-\pi;\pi) \text{je } \bar{f}(x) = f(x) \\ \bar{f}(x+2\pi) = \bar{f}(x) \text{ pro } x\epsilon(-\infty;+\infty) \end{cases}$$

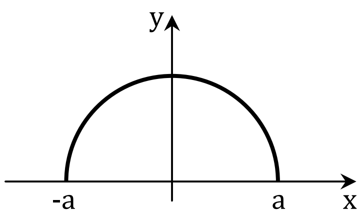
Př.

$$f(x) = \begin{matrix} x+\pi, x\epsilon(-\pi;0) \\ x, x\epsilon\langle 0;\pi) \end{matrix}$$



Sinová a kosinová Fourierova řada

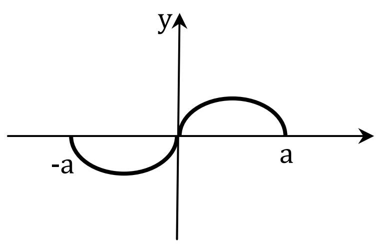
1.



f je sudá funkce v $\langle -a;a\rangle$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 2.\int_{0}^{a} f(x)dx$$

2.



f je lichá funkce v $\langle -a;a\rangle$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$