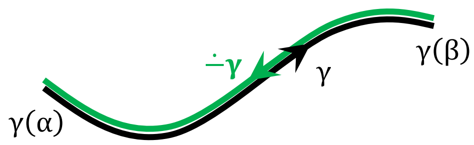
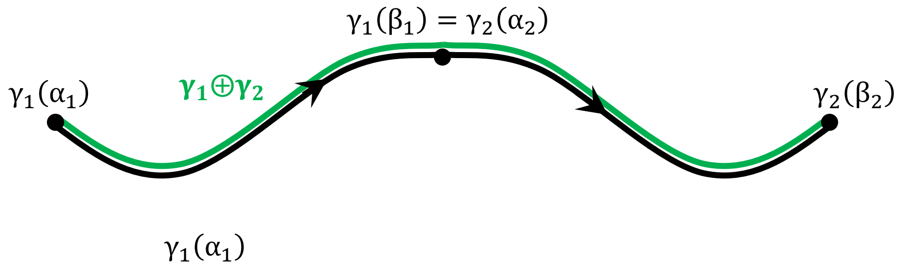
Operace s křivkami

1.



$\dot{-}\gamma$ – opačně orientovaná křivka ke γ

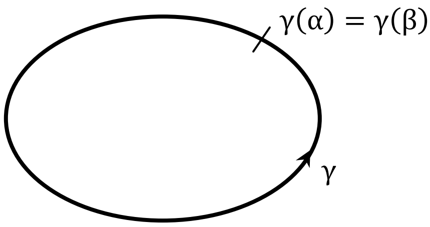
2.



– orientovaný součet křivek γ_1, γ_2

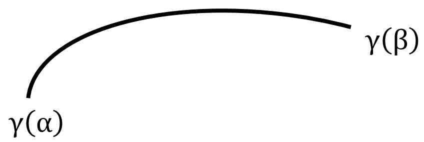
Vlastnosti křivek

1.



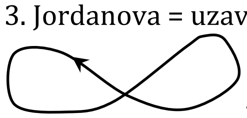
Uzavřená křivka

2. Jednoduchá křivka (prostá) sama sebe neprotíná



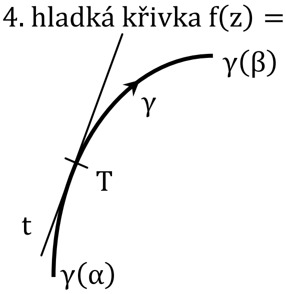
$\forall t_1, t_2 \epsilon (\alpha; \beta); \left(t_1 \neq t_2 \Rightarrow \gamma(t_1) \neq \gamma(t_2)\right)$

3. Jordanova = uzavřená + jednoduchá



– není Jordanova

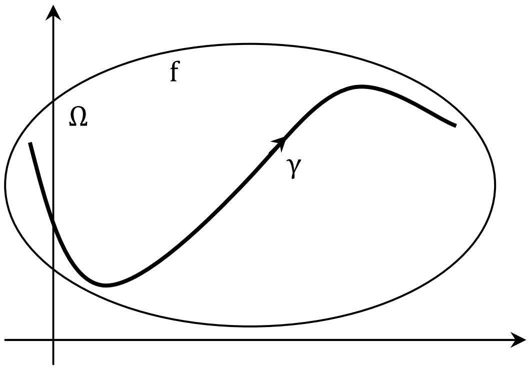
4. hladká křivka $f(z) = \gamma(t); t\epsilon\langle\alpha; \beta\rangle$



$\gamma(t)$ je spojitá a nezáporná funkce v $\langle\alpha; \beta\rangle$

5. Po částech hladká

Integrál funkce komplexní proměnné



jednoduchá po částech hladká křivka

f – spojitá (omezená)

součtová definice – **viz. skripta**

$$\int_{\gamma} f(z).\,dz$$

Křivkový integrál 2. druhu

Věta 4.22. $\gamma\langle\alpha;\beta\rangle \to C$ po částech hladká křivka, f je spojitá funkce na $[\gamma]$. Pak existuje $\int_{\gamma} f(z).\,dz$ a platí:

$$\int_{\gamma} f(z).\,dz = \int_{\gamma} (u + i.v).(dx + i.dy) = \int_{\gamma} u(x;y)dx - v(x;y)dy + i.\int_{\gamma} v(x;y)dx + u(x;y)dy$$

Věta 4.23. Po částech hladká křivka $z = \gamma(t) = x(t) + i.y(t)$, $t\epsilon\langle\alpha;\beta\rangle$.

$$\int_{\gamma} f(z).\,dz = \begin{vmatrix} x = x(t) & dx = x'(t)dt \\ y = y(t) & dy = y'(t)dt \end{vmatrix} =$$

$$= \int_{\alpha}^{\beta} \left[u\big(x(t);y(t)\big).x'(t) - v\left(x(t);y(t)\right).y'(t)\right].dt +$$

$$+ \int_{\alpha}^{\beta} \left[v\big(x(t);y(t)\big).x'(t) + u\big(x(t);y(t)\big)y'(t)\right].dt =$$

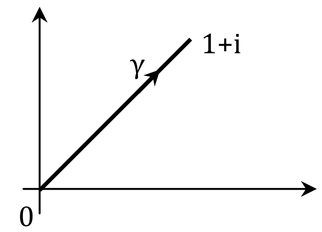
$$= \int_{\alpha}^{\beta} \left[u\big(x(t);y(t)\big) + i.v\big(x(t);y(t)\big)\right].\big(x'(t) + i.y'(t)\big).dt = \int_{\alpha}^{\beta} f\big(z(t)\big).dz =$$

$$= \int_{\alpha}^{\beta} f\big(\gamma(t)\big).\gamma'(t).dt$$

Př.

1.

$$\int_\gamma z.\,dz = \int_0^1 (t + i.\,t).\,(1 + i)dt = \int_0^1 0.\,dt + i.\int_0^1 2t.\,dt = i.\,[t^2]_0^1 = i$$



γ:

$x = t$

$y = t;\, t\epsilon\langle 0; 1\rangle$

$z(t) = t + i.\,t$

$dz = (1 + i)dt$

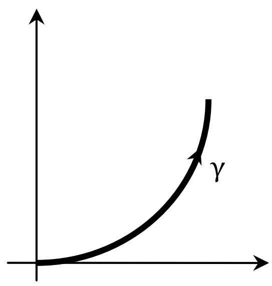
nebo:

$$(1 + i)^2 \int_0^1 t.\,dt = (1 + i)^2 \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}(1 + 2i + i^2) = i$$

2.

$$\int_\gamma z.\,dz = \int_0^1 (t + i.\,t^2).\,(1 + i2t).\,dt = \int_0^1 (t - 2t^3).\,dt + i.\int_0^1 (2t^2 + t^2).\,dt =$$

$$= \left[\frac{t^2}{2} - 2.\frac{t^4}{4}\right]_0^1 + i.\left[3.\frac{t^3}{3}\right]_0^1 = i$$



γ – parabola ($\gamma = x^2$)

$x = t$

$y = t^2$

γ:

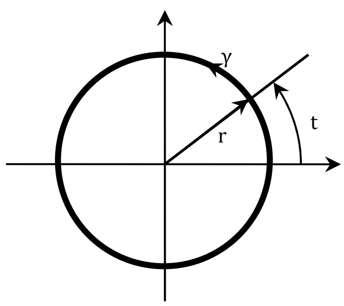
$z = t + i.\,t^2;\, t\epsilon\langle 0; 1\rangle$

$dz = (1 + i.\,2t)dt$

3. Důležitý !!!

$$\int_\gamma \frac{1}{z}.\,dz = \int_0^{2\pi} \frac{1}{r.\,(\cos t + i.\sin t)}.\,r.\,(-\sin t + i.\cos t).\,dt = \int_0^{2\pi} \frac{-\sin t + i.\cos t}{\cos t + i.\sin t}.\,dt =$$

$$= \int_0^{2\pi} \frac{-\sin t + i.\cos t}{\cos t + i.\sin t}.\frac{\cos t - i.\sin t}{\cos t - i.\sin t}.\,dt = i.\int_0^{2\pi} \frac{1}{\cos^2 t + \sin^2 t}.\,dt = i.\,[t]_0^{2\pi} = 2\pi i$$



$x = r.\cos t$

$y = r.\sin t$

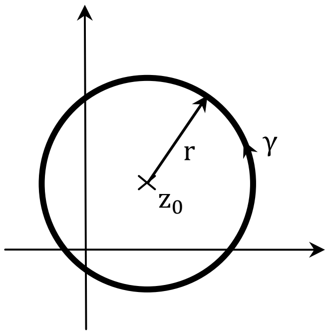
$\gamma(t) = r.\,(\cos t + i.\sin t);\;\; t\epsilon\langle 0; 2\pi\rangle$

$dz = r.\,(-\sin t + i.\cos t)dt$

$\gamma: z = z.\,e^{it}$

4.

$$\int_{\gamma} \frac{1}{z-z_0}.dz = \int_0^{2\pi} \frac{1}{r.e^{it}}.r.i.e^{it}.dt = i.\int_0^{2\pi} dt = 2\pi i$$



γ:

$z = z_0 + r.e^{it}$

$dz = r.i.e^{it}$

5.

$$\int_{\gamma} \frac{1}{(z-z_0)^n}.dz = \int_0^{2\pi} \frac{1}{(r.e^{it})^n}.r.i.e^{it} = \frac{i}{r^{n-1}}.\int_0^{2\pi} e^{i.(1-n).t}.dt = \frac{i}{r^{n-1}}.\left[\frac{e^{i.(1-n).t}}{i.(1-n)}\right]_0^{2\pi} =$$

$n \neq 1; n \epsilon Z$

$$= \frac{1}{(1-n).r^{n-1}}.\left(e^{i.(1-n).2\pi} - e^0\right) = 0$$

periodická funkce
s periodou $2\pi i$

Věta 4.24.

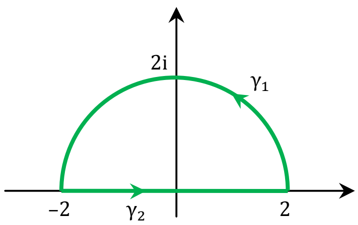
1. $\int_{\gamma} \big(a.f(z) + b.g(z)\big).dz = a.\int_{\gamma} f(z).dz + b.\int_{\gamma} g(z).dz -$ linearita integrálu

2. $\int_{\gamma_1 \oplus \gamma_2} f(z).dz = \int_{\gamma_1} f(z).dz + \int_{\gamma_2} f(z).dz -$ aditivita integrálu

3. $\int_{\dot{-}\gamma} f(z).dz = -\int_{\gamma} f(z).dz$

Př.

$$\int_{\gamma} \bar{z}.dz = \int_{\gamma_1 \oplus \gamma_2} \bar{z}.dz = \int_{\gamma_1} \bar{z}.dz + \int_{\gamma_2} \bar{z}.dz = 0 + i4\pi = 4\pi i$$



γ_1:

$z = t; t\epsilon\langle -2; 2\rangle$

γ_2:

$z = r.(\cos t + i.\sin t); z = e^{it}.2;\ t\epsilon\langle 0; \pi\rangle$

$$\boxed{\left|e^{it}\right| = |\cos t + i.\sin t| = \sqrt{\cos^2 t + \sin^2 t} = 1}$$

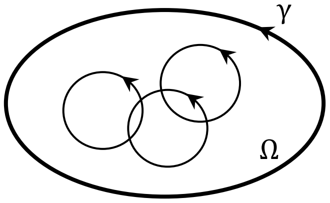
$$\int_{\gamma_1} = \int_{-2}^{2} t.dt = \left[\frac{t^2}{2}\right]_{-2}^{2} = 2 - 2 = 0$$

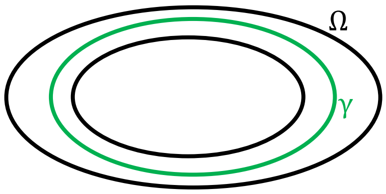
$$\int_{\gamma_2} = \int_0^{\pi} 2.e^{-it}.2.i.e^{it}.dt = 4i.\int_0^{\pi} dt = 4i.[t]_0^{\pi} = i4\pi$$

$\boxed{\bar{z} = 2.(\cos t - i.\sin t) = 2.(\cos(-t) + i.\sin(-t)) = 2.e^{-it}}$

Jednoduše souvislá oblast

$\Omega \Rightarrow$ vnitřek každé v ní obsažené Jordanovy křivky patří celá této oblasti (oblastí je jediná Jordanova křivka)





– dvojnásobně souvislá oblast

Věta 4.25. (Cauchova věta) Nechť f je holomorfní funkce na jednoduše souvislé oblasti Ω. Pak pro každou po částech hladkou Jordanovu křivku γ ležící v Ω platí:

$$\int_\gamma f(z).dz = 0$$

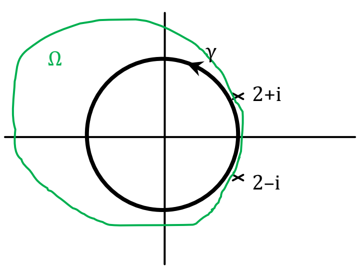
Důkaz: Pro křivkové integrály 2. druhu funkcí dvou reálných proměnných platí:

$\int_\gamma P(x;y).dx + Q(x;y).dy = 0$; jestliže $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ jsou spojité funkce v Ω a hlavně $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ v Ω.

Víme, že $\int_\gamma f(z).dz = \int_\gamma u(x;y).dx - v(x;y).dy + i.\int_\gamma v(x;y).dx + u(x;y).dy$. Podle C.–R. podmínek je $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ a $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, což ale znamená, že oba uvedené křivkové integrály se rovnají 0. Proto $\int_\gamma f(z).dz = 0$.

<u>Př.</u>

$$\int_\gamma \frac{z^2}{z^2 - 4z + 5}.dz = 0 \text{ podle Cauchyovy věty}$$



$\gamma: |z| = 2$

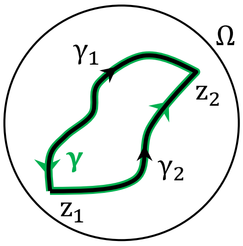
singulární body:

$z^2 - 4z + 5 = 0$

$z_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i$

Důsledky Cauchyovy věty

1. Nezávislost integrál na integrační cestě (**Věta 4.26.**)



– f je holomorfní funkce

$$\gamma_1 \oplus (\dot{-}\gamma_2) = \gamma$$

Podle Cauchyovy věty:

$$\int_{\gamma} f(z).\,dz = \int_{\gamma_1} f(z).\,dz + \int_{\dot{-}\gamma_2} f(z).\,dz = \int_{\gamma_1} f(z).\,dz - \int_{\gamma_2} f(z).\,dz = 0$$

Tzn. $\displaystyle\int_{\gamma_1} \quad = \int_{\gamma_2}$